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Archimedes' Squaring Of A Parabola

To determine the area enclosed in a parabola section

The squaring of a parabola is one of Archimedes' most remarkable achievements. It was accomplished about 240 b.C. and is based upon the properties of Archimedes triangle.
An Archimedes triangle is a triangle whose sides consist of two tangents to a parabola and the chord connecting the points of tangency. The last mentioned side is taken as the base line or the bas of the triangle. In order to construct such a triangle we draw the parallels to the parabola axis through the two points H and K of the directrix and erect the perpendicular bisectors upon the lines connecting H and K with the focus F. If we designate the point of intersection of the two perpendicular bisectors as S, the point of intersection of the first perpendicular bisector with the second parallel to the axis as B, then A and B are points of the parabola (classical construction of the parabola), and ASB is an Archimedes triangle.

Since SA and SB are two perpendicular bisectors of the triangle FHK, the parallel to the axis through S is the third perpendicular bisector; it consequently passes through the center of HK, and, as the midline of the trapezoid AHKB, it also passes through the center M of AB. This gives us the theorem: The median to the base of an Archimedes triangle is a parallel to the axis.
Let the parabola tangents through the point of intersection O of the median SM to the base with the parabola cult SA at A´, SB at B´. Then AA´O and BB´O are also Archimedes triangles. Consequently according to the above theorem, the medians to their bases are also

parallel to the axis and are therefore midlines in the triangles SAO and SBO, so that A´ and B´ are the centers of SA and SB. A´B´ is consequently the midline of the triangle SAB and is therefore parallel to AB; also the point O on A´B´ must be the center of SM.
The result of our investigations is the

Theorem of Archimedes: *the median to the base of an Archimedes triangle is parallel to the axis, the midline parallel to the base is a tangent, and its point of intersection with the median to the base is a point of the parabola.*

Now we can determine the area J of the parabola section enclosed in our Archimedes triangle ASB with the base line AB.
The tangents A´B´ and the chords OA and OB devide the triangle ASB into four sections: 1. The “internal triangle” AOB enclosed within the parabola; 2. The “external triangle” A´SB´ lying outside the parabola; 3. And 4. Two “residual triangles” AOA´ and BOB´, which are also Archimedes triangles and are penetrated by the parabola.
Since O lies at the center of SN, the internal triangle is twice the size of the external triangle.
In the same fashion, each of the two residual triangles intern gives rise to an internal triangle, an external triangle and two new residual Archimedes triangles that are penetrated by the parabola, and ones again each internal triangle is twice the size of the corresponding external triangle.
Thus, we can continue without end and cover the entire surface of the initial Archimedes triangle ASB with internal and external triangles. The sum of all internal triangles must also be twice as great as the sum of external triangles. In other words:

Theorem of Archimedes: *the parabola divides the Archimedes triangle into sections whose ratios 2:1.*

Or also:

The area enclosed by a parabola section is two thirds the area of the corresponding Archimedes triangle.

Archimedes arrived at this conclusion by a somewhat different method. He found the area of the section by adding together the areas of all the successive internal triangles.
If Δ represents the area of the initial Archimedes triangles ASB, then the area of the corresponding internal is one half Δ, the area of the corresponding external triangle is one quarter of Δ, and the area of each of the two residual triangles is one eighth of Δ.
The successive Archimedes triangles therefore have the areas

$$\Delta, \frac{\Delta}{8}, \frac{\Delta}{8^2}, \ldots;$$

the corresponding internal triangles possess half this area; and since each internal triangle gives rise to *two* new internal triangles, we thus obtain for the sum of all the successive internal triangle areas the value.

$$\frac{1}{2}\left[\Delta + 2\frac{\Delta}{8} + 4\frac{\Delta}{8^2} + 8\frac{\Delta}{8^3} + \ldots\right]$$

The bracket encloses a geometrical series with the quotient ¼, the sum of which is equal to $\Delta / \left(1 - \frac{1}{4}\right) = \frac{4}{3}\Delta$. Thus we again obtain for the area of the section the value $J = \frac{2}{3}\Delta$.

Since A´B´ is tangent to the parabola at O, the perpendicular h dropped from O to the base line AB of the section is the altitude of the section. Since h is also half the altitude of the triangle ASB, $\Delta = AB*h$ and $J = \frac{2}{3}*AB*h$, i.e.:

The area enclosed by a parabola section is equal to two thirds the product of the base and the altitude of the section.

Finally, we will express the area of the section in terms of the transverse q of the section, i.e., by the projection normal to the axis of the chord bounding the section.

We use the equation for the amplitude of the parabola, calling the coordinates of the corners of the section $x|y$ and $X|Y$, and we have

$$y^2 = 2px \text{ and } Y^2 = 2pX$$

with 2p representing the parameter. From the given figure it follows directly that

$$J = \frac{2}{3}XY - \frac{2}{3}xy - (X - x)\frac{Y + y}{2}.$$

If we replace X and x here with $Y^2/2p$ and $y^2/2p$, we obtain $12pJ=Y^3-y^3-3Y^2y+3Yy^2=(Y-y)^3$. Since Y-y is the section transverse q, we finally obtain

$$12pJ=q^3.$$

This important formula can be expressed verbally as follows:

Six times the product of the parameter and the area of the section is equal to the cube of the section transverse.